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VECTORS FOR BEGINNERS

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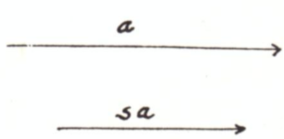
The vector idea is so common in the study of science and the vector method so easy and effectual in application that students in preparatory schools should acquire an understanding of the subject. Its importance is emphasized when we recall that displacement, velocity, acceleration, force, electric current, stresses, strains and many other physical quantities can be correctly represented by vectors.

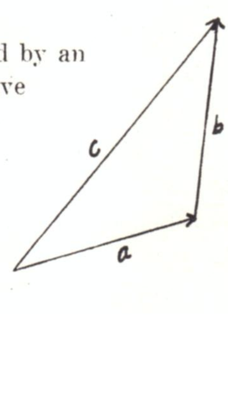
A vector is a quantity that has magnitude and direction and can therefore be represented by a directed segment of a straight line. A scalar quantity lacks the quality of direction and ordinarily may be considered as of the nature of real number varying from $-\infty$ through zero to $+\infty$ like the numbers on a linear scale which proceed either negatively or positively from the zero point.



If we represent a vector **a** by a directed segment then s**a**, s being a scalar quantity is a vector parallel to **a** which is less than **a** if $0 < s < 1$ and greater than **a** if $s > 1$. If s is negative, the vector s**a** is then directed in the opposite direction from that of **a**.

If c is the third side of a triangle, the other sides being a and b directed as shown, c is said to be the vector sum of a and b or

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$



If we think of the vector **b** as being replaced by an equal and opposite vector $-$**b** we should have

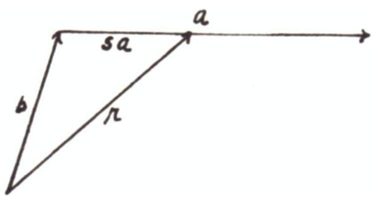
$$\mathbf{c} + (-\mathbf{b}) = \mathbf{a} \quad \text{or} \quad \mathbf{c} - \mathbf{b} = \mathbf{a},$$

that is, we may transpose as in an ordinary algebraic equation or subtract vectors if we pay due regard to the direction of the vectors.

Again, if we have the vector equation

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{d}$$

one can easily verify that we have the same result whether we add $\mathbf{a}$ to the sum of $\mathbf{b}$ and $\mathbf{c}$ or $\mathbf{b}$ to the sum of $\mathbf{a}$ and $\mathbf{c}$ or $\mathbf{c}$ to the sum of $\mathbf{a}$ and $\mathbf{b}$; that is the commutative and associative laws of algebraic addition hold.



In the equation

$$\mathbf{r} = \mathbf{b} + s\mathbf{a}$$

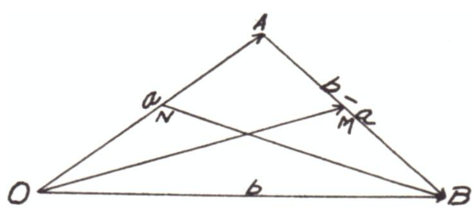
$\mathbf{r}$ is a vector from the origin of $\mathbf{b}$ to any point on $\mathbf{a}$ or $\mathbf{a}$ extended, depending upon the value of s.

It is evident that the sum of two non-parallel vectors cannot be zero unless each vector is zero, so that, if

$$(s' - s)\mathbf{a} + (t' - t)\mathbf{b} = 0$$

s, s', t and t' being scalars we must have $s - s' = 0$ and $t - t' = 0$ or $s = s'$ and $t = t'$.

This is sufficient knowledge to begin making applications to problems in geometry. To prove that the medians of a triangle intersect in a point two-thirds the distance from each vertex to the opposite side we assume $\mathbf{a}$ and $\mathbf{b}$ as vectors for the sides OA and OB of the triangle. Then $\mathbf{b} - \mathbf{a}$ is the third side $\overrightarrow{AB}$, that is directed from A to B. Then the median $\overrightarrow{OM}$ is



$$\overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Hence for the vector from O to any point on OM we have

$$\mathbf{r} = \tfrac{1}{2}s(\mathbf{a} + \mathbf{b})$$

For the median $\overrightarrow{NB}$ we have $\overrightarrow{ON} + \overrightarrow{NB} = \overrightarrow{OB}$ or $\overrightarrow{NB} = \mathbf{b} - \tfrac{1}{2}\mathbf{a}$ and since N is the origin of this vector we have with O as origin for any point on it.

$$\mathbf{r} = \overrightarrow{ON} + t(\overrightarrow{NB}) = \tfrac{1}{2}\mathbf{a} + t(\mathbf{b} - \tfrac{1}{2}\mathbf{a})$$

where the vectors $\overrightarrow{OM}$ and $\overrightarrow{NB}$ intersect the **r**'s are identical and therefore

$$\tfrac{1}{2}s(\mathbf{a}+\mathbf{b}) = \tfrac{1}{2}\mathbf{a} + t(\mathbf{b} - \tfrac{1}{2}\mathbf{a})$$

or

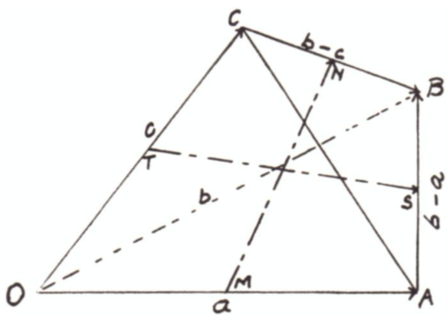
$$\tfrac{1}{2}(s + t - 1)\,\mathbf{a} + (\tfrac{1}{2}s - t)\,\mathbf{b} = 0$$

Now, since the coefficients of **a** and **b** must vanish

$$s + t - 1 = 0 \text{ and } \tfrac{1}{2}s - t = 0$$

giving $t = \frac{1}{3}$ and $s = \frac{2}{3}$; that is the medians intersect two-thirds the distance from O to M, and one-third the distance from N to B.

Again, let us take the proposition from solid geometry that the lines from the middle points of opposite sides of a tetraedron meet and bisect each other.



In the tetraedron $OABC$ let $\overrightarrow{OA}$ be **a**, $\overrightarrow{OB}$ be **b**, $\overrightarrow{OC}$ be **c**, then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{CB} = \mathbf{b} - \mathbf{c}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$.

For the vector $\overrightarrow{MN}$ from the middle of OA to the middle of CB we have

$$\tfrac{1}{2}\mathbf{a} + \overrightarrow{MN} = \mathbf{c} + \tfrac{1}{2}(\mathbf{b} - \mathbf{c})$$

whence

$$\overrightarrow{MN} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{a})$$

and for any point on MN we have, O being the origin

$$\mathbf{r} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}s(\mathbf{b} + \mathbf{c} - \mathbf{a})$$

For the vector $\overrightarrow{TS}$ from the middle of OC to the middle of AB we have

$$\tfrac{1}{2}\mathbf{c} + \overrightarrow{TS} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a})$$

giving

$$\overrightarrow{TS} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

so for any point on TS, O being origin

$$\mathbf{r} = \tfrac{1}{2}\mathbf{c} + \tfrac{1}{2}t(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

s and t being scalar quantities. If these intersect

$$\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}s(\mathbf{b} + \mathbf{c} - \mathbf{a}) = \tfrac{1}{2}\mathbf{c} + \tfrac{1}{2}t(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

and it must be possible for s and t to satisfy the three equations:

$$1 - s - t = 0 \qquad s - 1 + t = 0 \qquad s - t = 0$$

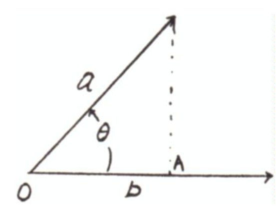
which can be done by $s = \frac{1}{2}$ and $t = \frac{1}{2}$.

The two vectors therefore intersect and at their middle points. The ease and directness of this method will certainly appeal to any pupil who has the elements of a real student in him.

The absolute length of a vector $\mathbf{a}_o$, usually written $\mathbf{a}_o$, is its magnitude without regard to its direction. The absolute length of the vector 5 miles northwest is 5 miles.

The *dot* or *scalar* product of two vectors (written $a \cdot b$) is a scalar quantity equal to the product of the absolute values of one vector times the projection of the other upon it. Thus:

$$\mathbf{a} \cdot \mathbf{b} = (OA)\mathbf{b}_o = \mathbf{a}_o\mathbf{b}_o cos\ \theta$$



where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

If θ is zero we have, letting $\mathbf{b} = \mathbf{a}$,

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{a}_o{}^2 = a^2$$

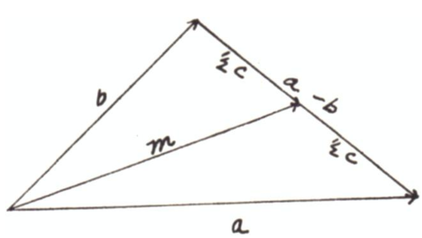
so we get the important relation

$$a = \mathbf{a}_o = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

or the absolute length of a vector is the square root of the dot product of the vector by itself. Dot products follow the ordinary distributative and commutative laws of multiplication, that is:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$$

As an application of the dot product we take the proposition from the third book of plane geometry that the square of the median of a triangle equals half the sum of the squares of the two including sides less the square of half the third side.



Let the sides of the triangle be of lengths a, b and c and the median m as shown in the figure. Then we have

$$\mathbf{m} = \mathbf{b} + \tfrac{1}{2}(\mathbf{a} - \mathbf{b}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\mathbf{m}\cdot\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})\cdot\tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$= \tfrac{1}{4}(\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b})$$

or

$$m^2 = \tfrac{1}{4}(a^2 + b^2) + \tfrac{1}{2}\mathbf{a}\cdot\mathbf{b}$$

Now

$$\mathbf{b} = \mathbf{m} - \tfrac{1}{2}\mathbf{c}$$

$$\mathbf{a} = \mathbf{m} + \tfrac{1}{2}\mathbf{c}$$

$$\therefore \mathbf{a}\cdot\mathbf{b} = (\mathbf{m} + \tfrac{1}{2}\mathbf{c})\cdot(\mathbf{m} - \tfrac{1}{2}\mathbf{c}) = \mathbf{m}\cdot\mathbf{m} - \tfrac{1}{4}\mathbf{c}\cdot\mathbf{c} = m^2 - \tfrac{1}{4}c^2$$

so that substituting this value we have

$$m^2 = \tfrac{1}{4}(a^2 + b^2) + \tfrac{1}{2}m^2 - \tfrac{1}{8}c^2$$

or

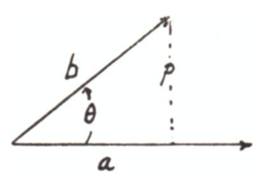
$$m^2 = \tfrac{1}{2}(a^2 + b^2) - (\tfrac{1}{2}c)^2$$

Q. E. D.

Obviously, many problems and theorems involving length of lines can be treated in this way.

The *cross* or *vector product* of two vectors (written $\mathbf{a} \times \mathbf{b}$) is a vector quantity of absolute magnitude equal to the product of the absolute value of $\mathbf{a}$ and of the projection of $\mathbf{b}$ upon a perpendicular to $\mathbf{a}$ and of direction perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Thus:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a}_o p\mathbf{k} = \mathbf{a}_o\mathbf{b}_o sin\theta\mathbf{k}$$



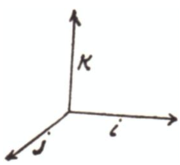
where $\mathbf{k}$ is a vector of unit length perpendicular to the paper downward or in a direction opposite that of the motion of a right-handed screw if a line across its head be turned across the angle θ from the direction of $\mathbf{a}$ to that of $\mathbf{b}$. We have then, that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

that is, if we change the order of the factors we change the sign of the cross product. Again $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is not equal to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ so that the commutative and associate laws of algebraic multiplication do not hold in the case of the cross product. But, the order remaining the same, the distributive law of algebraic multiplication does hold, thus:

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} = -(\mathbf{b}+\mathbf{c})\times\mathbf{a} = -\mathbf{b}\times\mathbf{a}-\mathbf{c}\times\mathbf{a}$$

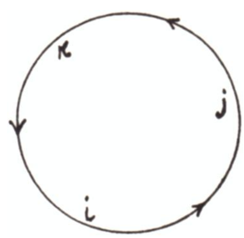
Many problems are simplified by using for reference three mutually perpendicular unit vectors **i**, **j**, and **k**. In this case since $\theta = 90°$, we have



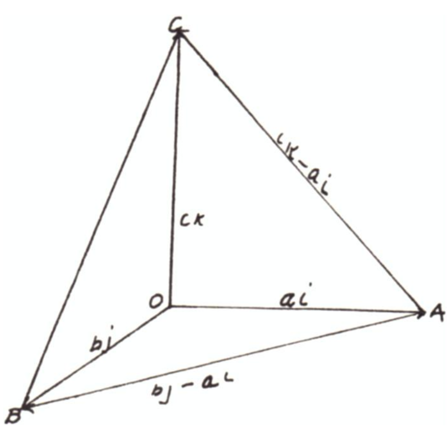
$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$
$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$
$$\mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k}$$
$$\mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j} = \mathbf{i}$$
$$\mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k} = \mathbf{j}$$

The last three of these results can be read cyclicly from a diagram. In the direction of the arrow heads we get positive products, the other way negative, *i. e.*,

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \text{ but } \mathbf{k}\times\mathbf{j} = -\mathbf{i}.$$



Since the absolute value of the cross product is equal to the area of the parallelogram whose sides are a and b, it is often useful in problems or theorems involving areas. From page 329 of Chauvenet's *Solid Geometry* we have the original: If one of the triedral angles of a tetraedron is trirectangular the square of the area of the face opposite to it is equal to the sum of the squares of the areas of the three other faces.



Let the three mutually perpendicular edges be of length a, b and c, then using unit vectors we have $\overrightarrow{OA} = a\mathbf{i}$, $\overrightarrow{OB} = b\mathbf{j}$ and $\overrightarrow{OC} = c\mathbf{k}$, and for the edges AB and AC opposite the trirectangular angle

$$\overrightarrow{AB} = b\mathbf{j} - a\mathbf{i};$$

$$\overrightarrow{AC} = c\mathbf{k} - a\mathbf{i}.$$

Then

$$
\begin{aligned}
2 \text{ Area } \triangle ABC &= [(b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i})]_o \\
&= [bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}]_o \\
&= \sqrt{(bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}) \cdot (bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k})} \\
&= \sqrt{b^2c^2 + a^2c^2 + a^2b^2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\text{Area } ABC)^2 &= (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ac)^2 + (\tfrac{1}{2}ab)^2 \\
&= (\text{Area } OBC)^2 + (\text{Area } OAC)^2 + (\text{Area } OAB)^2
\end{aligned}
$$

Q. E. D.

The value of the principles here set forth is more real than is apparent; for the student having a thorough grasp of them can not only solve many problems in elementary mathematics, but is equipped to pass by easy stages through most of the analyses of analytic geometry, differential and integral calculus, analytic mechanics and differential geometry besides much of physics and electrical theory.